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LECTURE 18

Graphs – EULER & HAMILTON

INTRODUCTION TO PATHS

- Many problems can be modeled with paths formed by traveling along the edges of graphs.
- For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model.
- Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on can be solved using models that involve paths in graphs.

WHAT IS A PATH?

- Informally, a **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.
- As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these edges.

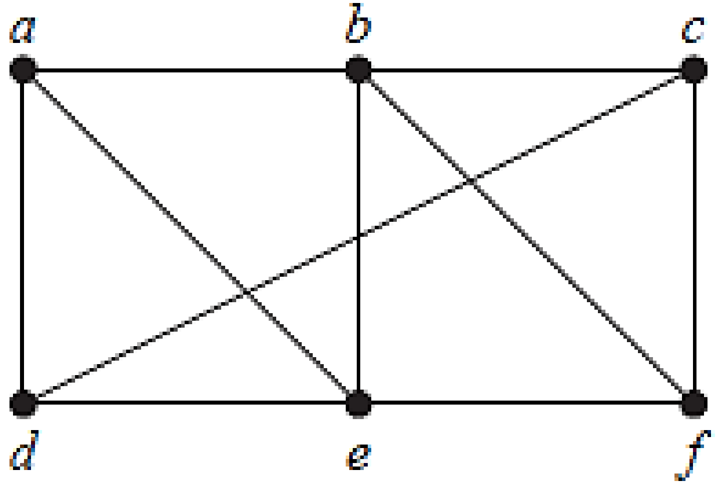
Let n be a nonnegative integer and G an undirected graph. A *path* of *length* n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots, n$, the endpoints x_{i-1} and x_i. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ (because listing these vertices uniquely determines the path). The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ or *traverse* the edges $e_1, e_2, \ldots, e_n$. A path or circuit is *simple* if it does not contain the same edge more than once.

DEFINITION OF A PATH

Let n be a nonnegative integer and G a directed graph. A *path* of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a *circuit* or *cycle*. A path or circuit is called *simple* if it does not contain the same edge more than once.

EXAMPLE 1

- In the simple graph shown in Figure below; *a*, *d*, *c*, *f*, *e* is a **simple path** of length 4, because {*a*, *d*}, {*d*, *c*}, {*c*, *f* }, and {*f*, *e*} are all edges. However, *d*, *e*, *c*, *a* is **not a path**, because {*e*, *c*} is not an edge. Note that *b*, *c*, *f*, *e*, *b* is a **simple circuit** of length 4 because {*b*, *c*}, {*c*, *f* }, {*f*, *e*}, and {*e*, *b*} are edges, and this path begins and ends at *b*. The path *a*, *b*, *e*, *d*, *a*, *b*, which is of length 5, is **not simple** because it contains the edge {*a*, *b*} twice.



EULER & HAMILTON PATHS

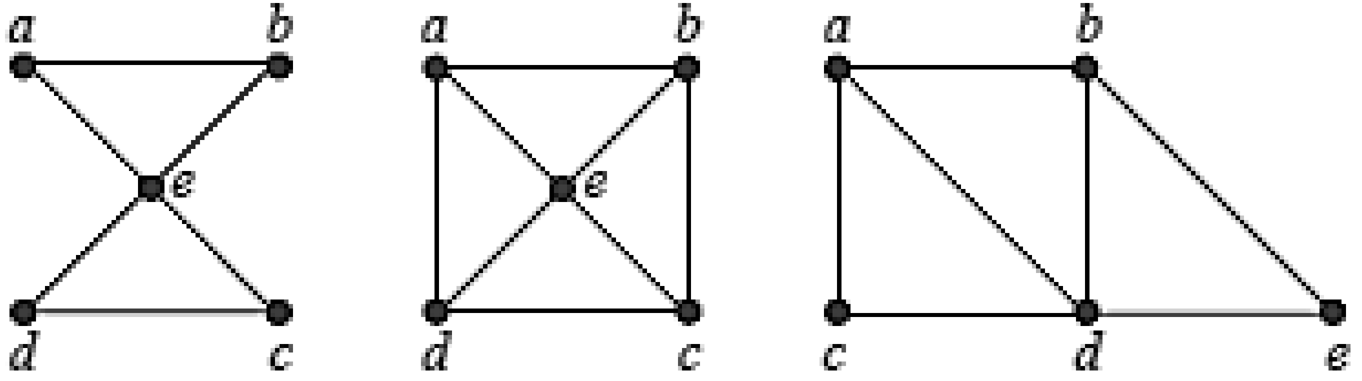
- Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each **edge** of the graph exactly once?
- Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each **vertex** of the graph exactly once?
- Although these questions seem to be similar, the first question, which asks whether a graph has an ***Euler circuit***.
- while the second question, which asks whether a graph has a ***Hamilton circuit.***

EULER PATHS & CIRCUITS

- An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.
- An *Euler path* in *G* is a simple path containing every edge of *G*.
- An **Euler circuit** starts and ends at **the same** vertex.
- An **Euler path** starts and ends at **different** vertex.
- A connected multigraph with at least two vertices has an Euler circuit if and only if **each of its vertices has even degree**.
- A connected multigraph has an Euler path but not an Euler circuit if and only if it has **exactly two vertices of odd degree**.

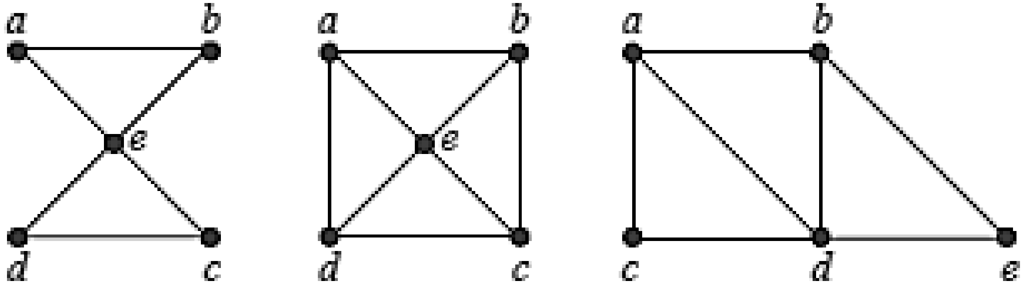
EXAMPLE 2

- Which of the undirected graphs in Figure below have an Euler circuit? Of those that do not, which have an Euler path?



EXAMPLE 2

- Which of the undirected graphs in Figure below have an Euler circuit? Of those that do not, which have an Euler path?



The graph *G*1 has an Euler circuit, for example, *a, e, c, d, e, b, a*.
Neither of the graphs *G*2 or *G*3 has an Euler circuit.
However, *G*3 has an Euler path, namely, *a, c, d, e, b, d, a, b*.
*G*2 does not have an Euler path.

EXAMPLE 3

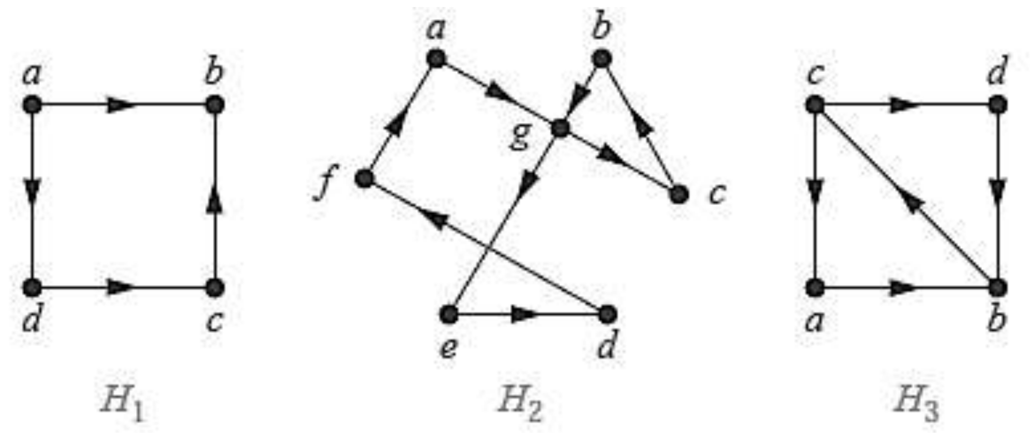
- Which of the directed graphs in Figure below have an Euler circuit? Of those that do not, which have an Euler path?

Solution:

The graph *H*2 has an Euler circuit, for example, *a, g, c, b, g, e, d, f, a.*

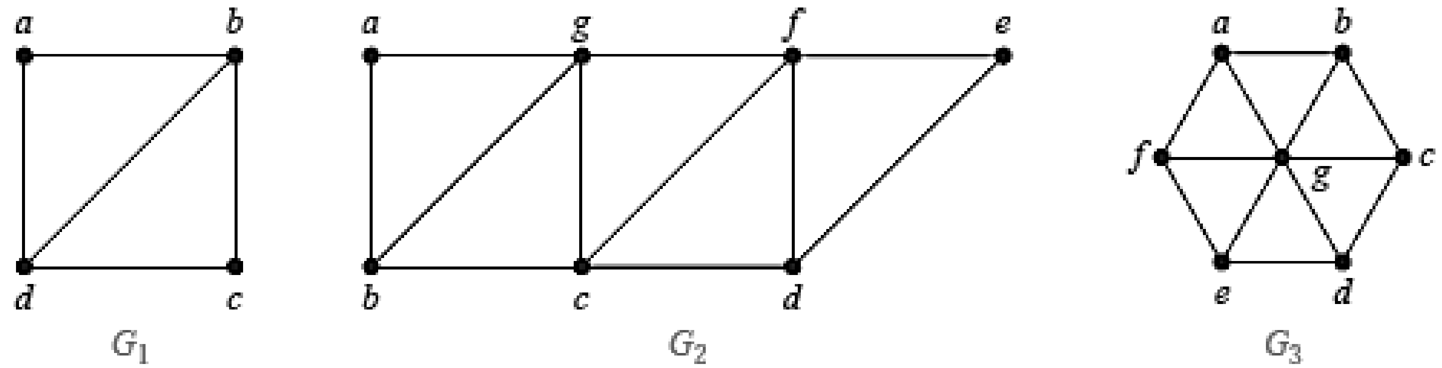
Neither *H*1 nor *H*3 has an Euler circuit.

*H*3 has an Euler path, namely, *c, a, b, c, d, b*, but *H*1 does not.



EXAMPLE 4

- Which graphs shown in Figure below have an Euler path?



*G*1 contains exactly two vertices of odd degree, namely, *b* and *d*. Hence, it has an Euler path that must have *b* and *d* as its endpoints. One such Euler path is *d, a, b, c, d, b*. Similarly, *G*2 has exactly two vertices of odd degree, namely, *b* and *d*. So it has an Euler path that must have *b* and *d* as endpoints. One such Euler path is *b, a, g, f, e, d, c, g, b, c, f, d*.

*G*3 has no Euler path because it has six vertices of odd degree.

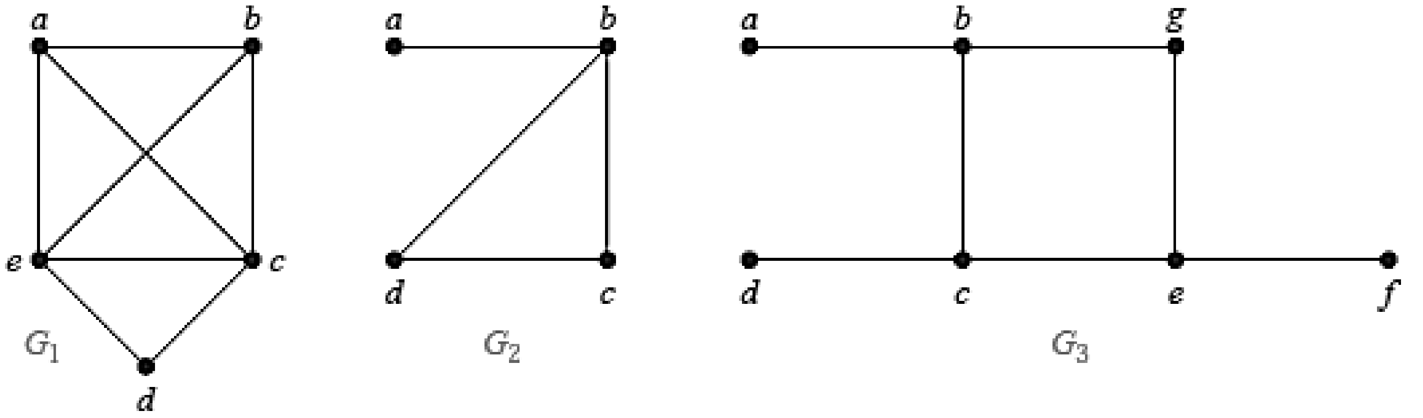
HAMILTON PATHS & CIRCUITS

A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

- There are no known simple necessary and sufficient criteria for the existence of Hamilton circuits. However, many theorems are known that give sufficient conditions for the existence of Hamilton circuits. Also, certain properties can be used to show that a graph has no Hamilton circuit.
- For instance, a graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the circuit. Moreover, if a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.
- a Hamilton circuit cannot contain a smaller circuit within it.

EXAMPLE 5

- Which of the simple graphs in Figure have a Hamilton circuit or, if not, a Hamilton path?



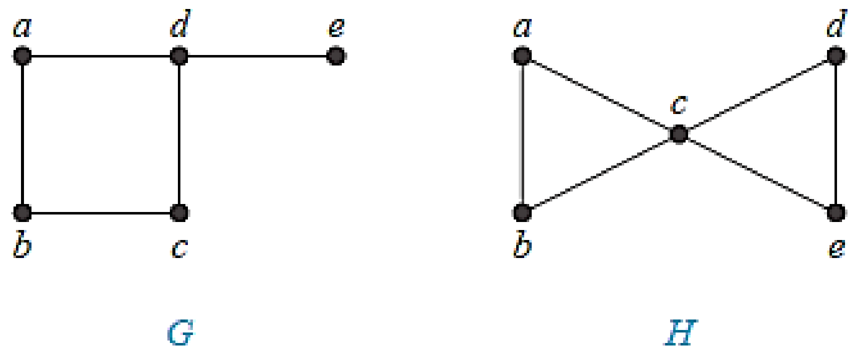
*G*1 has a Hamilton circuit: *a, b, c, d, e, a*.

There is no Hamilton circuit in *G*2 (this can be seen by noting that any circuit containing every vertex must contain the edge {*a, b*} twice), but *G*2 does have a Hamilton path, namely, *a, b, c, d*.

*G*3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges {*a, b*}, {*e, f* }, and {*c, d*} more than once.

EXAMPLE 6

- Show that neither graph displayed in Figure has a Hamilton circuit.



There is no Hamilton circuit in *G* because *G* has a vertex of degree one, namely, *e*.
Now consider *H*. It is easy to see that no Hamilton circuit can exist in *H*, for any Hamilton circuit would have to contain four edges incident with *c*, which is impossible.

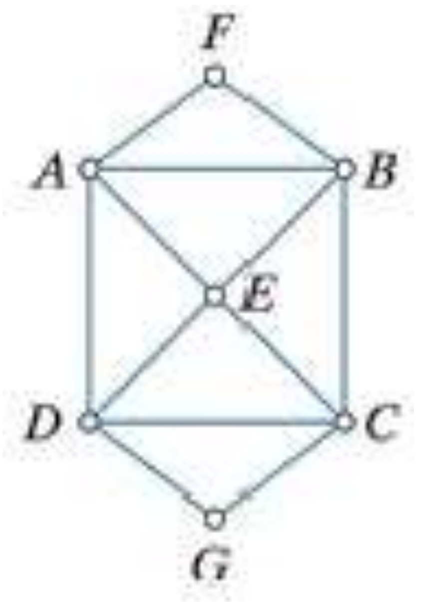
HAMILTON VERSUS EULER

EXAMPLE 7

- The figure shows a graph that has
- **Euler circuits** (the vertices are all even)

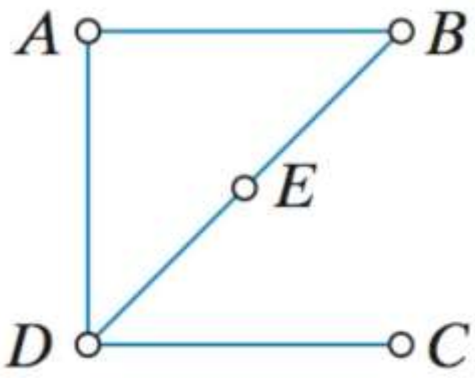
e.g. b, f, a, b, e, a, d, e, c, d, g, c, b and

- **Hamilton circuits** e.g. e, c, g, d, a, f, b, e.



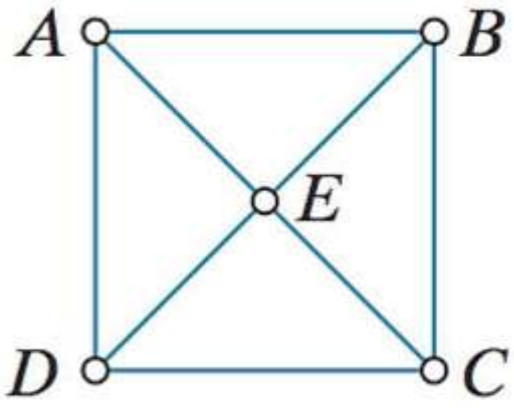
EXAMPLE 8

- The figure shows a graph that
- has **no Euler circuits** but does have **Euler paths** (for example C, D, E, B, A, D).
- It has **no Hamilton circuits** (sooner or later you have to go to C, and then you are stuck) but does have **Hamilton paths** (for example, A, B, E, D, C).



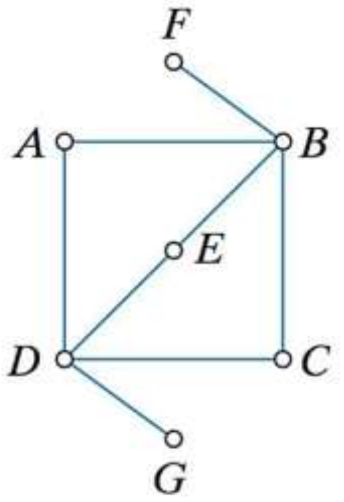
EXAMPLE 9

- The figure shows a graph that
- has **neither Euler circuits nor paths** (it has four odd vertices) and
- has **Hamilton circuits** (for example A, B, C, D, E, A - there are plenty more)



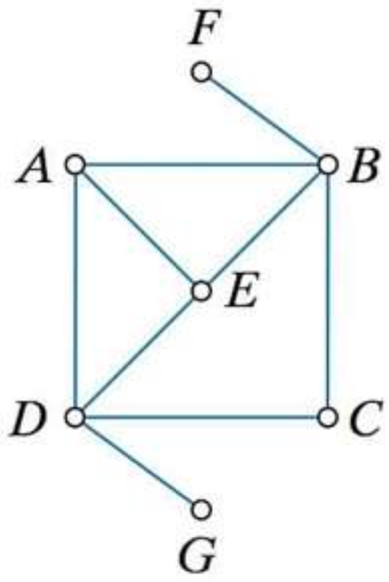
EXAMPLE 10

- The figure shows a graph that
- has **no Euler circuits** but has **Euler paths** (F and G are the two odd vertices) and
- has **neither Hamilton circuits nor Hamilton paths**.



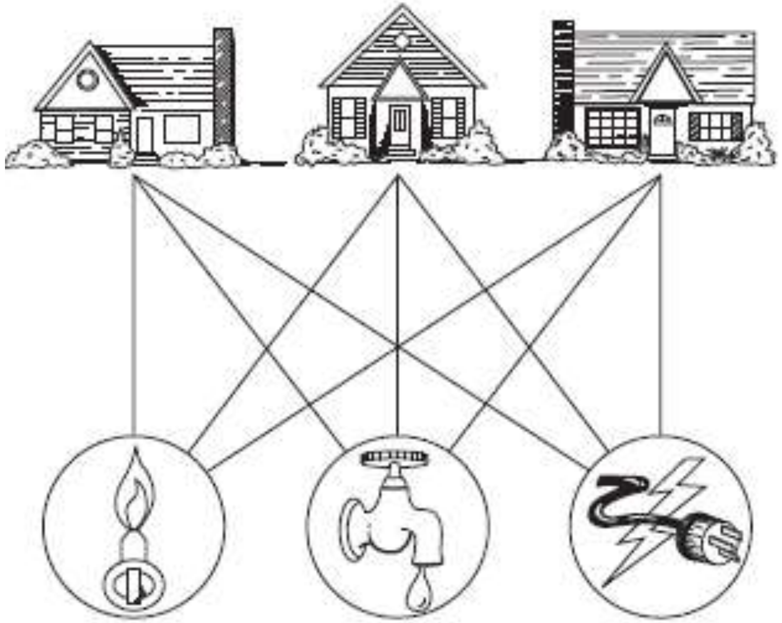
EXAMPLE 11

- The figure shows a graph that
- has neither Euler circuits nor Euler paths (too many odd vertices) and
- has neither Hamilton circuits nor Hamilton paths.



PLANAR GRAPH

- Consider the problem of joining three houses to each of three separate utilities, as shown in Figure. Is it possible to join these houses and utilities so that none of the connections cross?
- **In this section we will study the question of whether a graph can be drawn in the plane without edges crossing. In particular, we will answer the houses-and-utilities problem.**
- There are always many ways to represent a graph. When is it possible to find at least one way to represent this graph in a plane without any edges crossing?

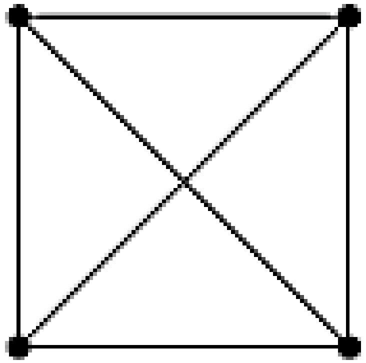


PLANAR GRAPH

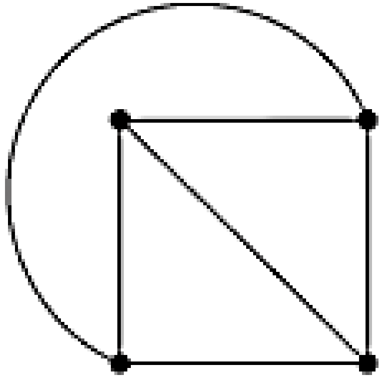
- A graph is called *planar* if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a *planar representation* of the graph.
- A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

EXAMPLE 12

- Is K_4 shown in Figure (with two edges crossing) planar?

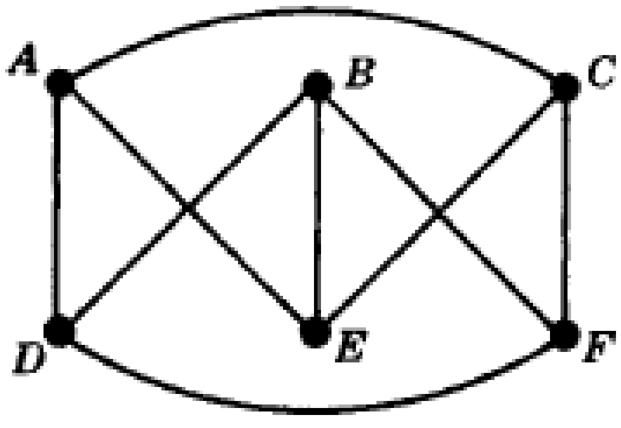


- K_4 is planar because it can be drawn without crossings.



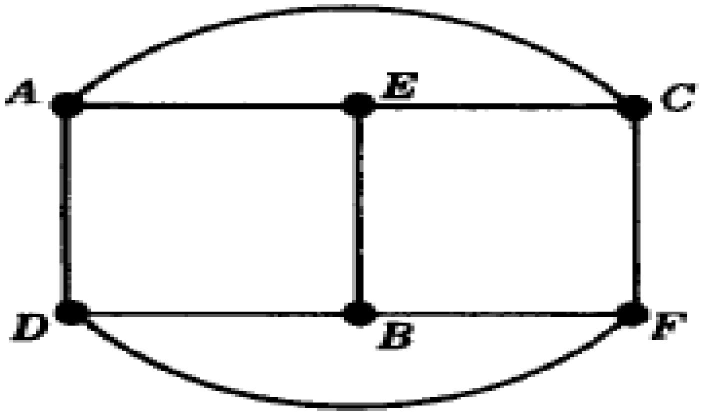
EXAMPLE 13

- Draw a planar representation, if possible.



(a)

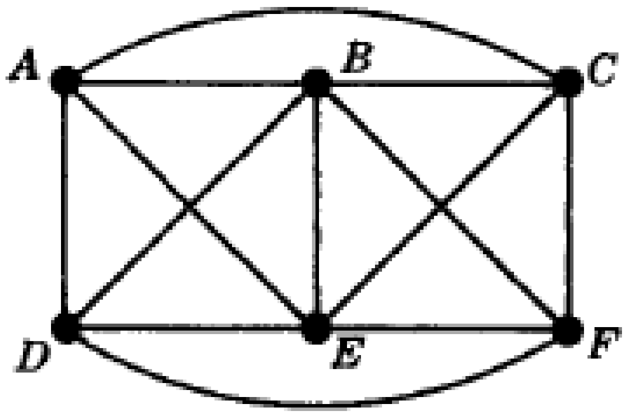
- Redrawing the positions of *B* and *E*, we get a planar representation of the graph as in Figure.



(a)

EXAMPLE 14

- Draw a planar representation, if possible.



(c)

Not possible because this graph is non-planar.

APPLICATIONS OF PLANAR GRAPHS

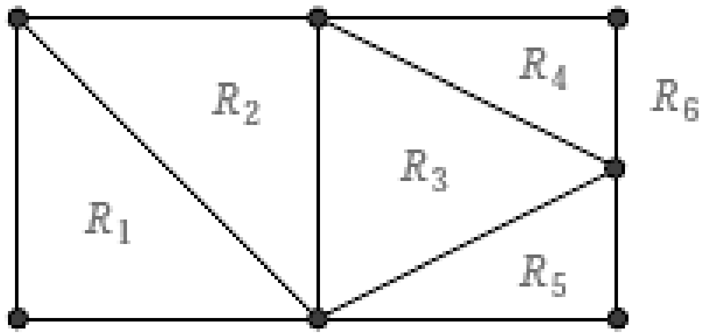
- Planarity of graphs plays an important role in the **design of electronic circuits**. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar. When this graph is not planar, we must turn to more expensive options. For example, we can partition the vertices in the graph representing the circuit into planar subgraphs. We then construct the circuit using multiple layers. We can construct the circuit using insulated wires whenever connections cross. In this case, drawing the graph with the fewest possible crossings is important.
- The planarity of graphs is also useful in the **design of road networks**. Suppose we want to connect a group of cities by roads. We can model a road network connecting these cities using a simple graph with vertices representing the cities and edges representing the highways connecting them. We can built this road network without using underpasses or overpasses if the resulting graph is planar.

EULER'S FORMULA

- A planar representation of a graph splits the plane into regions, including an unbounded region.
- For instance, the planar representation of the graph shown in Figure splits the plane into six regions as labeled.
- Euler showed that all planar representations of a graph split the plane into the same number of regions. He accomplished this by finding a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.

- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then

$$r = e - v + 2 \quad \text{or} \quad v - e + r = 2$$



EXAMPLE 15

- Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

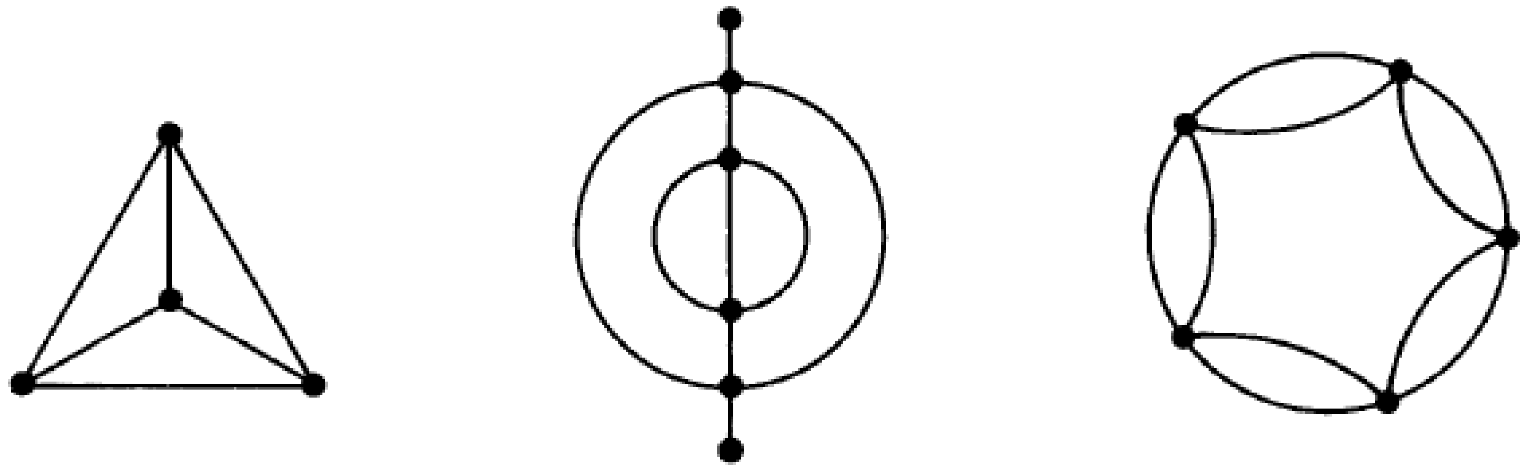
Solution:

- This graph has 20 vertices, each of degree 3, so $v = 20$.
- Because the sum of the degrees of the vertices, $3v = 3 \cdot 20 = 60$, is equal to twice the number of edges, $2e$, we have $2e = 60$, or $e = 30$.
- Consequently, from Euler's formula, the number of regions is

$$r = e - v + 2 = 30 - 20 + 2 = 12.$$

EXAMPLE 16

- Count the number V of vertices, the number E of edges, and the number R of regions of the graphs below and verify Euler's formula.

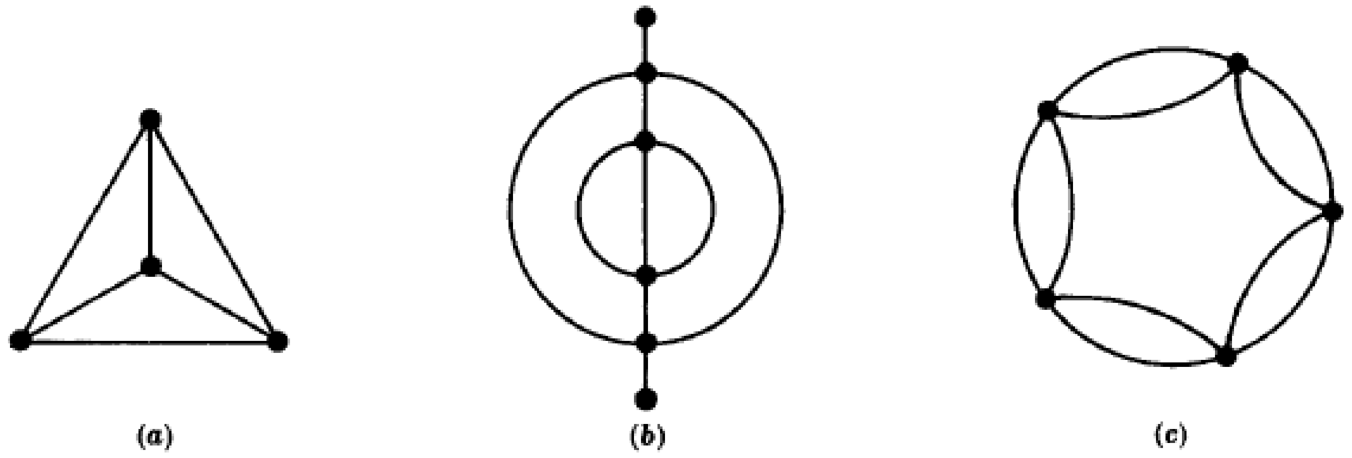


(a) (b) (c)

EXAMPLE 16

- Count the number V of vertices, the number E of edges, and the number R of regions of the graphs below and verify Euler's formula. ($R = E - V + 2$ *i.e.* $V - E + R = 2$

(a) $V = 4$, $E = 6$, $R = 4$. Hence $V - E + R = 4 - 6 + 4 = 2$.
(b) $V = 6$, $E = 9$, $R = 5$. Hence $V - E + R = 6 - 9 + 5 = 2$.
(c) $V = 5$, $E = 10$, $R = 7$. Hence $V - E + R = 5 - 10 + 7 = 2$.



APPLICATIONS OF HAMILTON CIRCUITS

- Hamilton paths and circuits can be used to solve practical problems.
- For example, many applications ask for a path or circuit that visits each road intersection in a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once.
- Finding a Hamilton path or circuit in the appropriate graph model can solve such problems. The famous traveling salesperson problem or TSP (also known in older literature as the traveling salesman problem) asks for the shortest route a traveling salesperson should take to visit a set of cities.

APPLICATIONS OF EULER PATHS AND CIRCUITS

- Euler paths and circuits can be used to solve many practical problems. For example:
- Many applications ask for a path or circuit that traverses each street in a neighborhood, each ***road in a transportation network***, each connection in a utility grid, or each link in a communications network exactly once.
- Among the other areas where Euler circuits and paths are applied is:
- The ***layout of circuits***, in network multicasting, and in ***molecular biology***, where Euler paths are used in the sequencing of DNA.

REFERENCES

- Kenneth Rosen Discrete Mathematics and Its Applications – Chapter # 10